Digital implementation of Hilbert Transform in the LCT domain associated with FIR filter

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***Abstract*— In this paper, digital implementation of Hilbert Transform in the LCT domain is proposed based on FIR filter. First, some definitions are described such as linear canonical transform, Hilbert transform, and analytical signal. Then, the implementation principle is analyzed about Hilbert transform in the LCT domain. Finally, the digital implementation method is proposed based on FIR filter.**

I. INTRODUCTION

The classic Hilbert transform (HT) has been playing an important role in the theory and practice of signal processing [1-3]. In recent years, the HT was generalized to some fractional versions, which is called fractional HT [4-7].

Linear canonical transform (LCT) has been proposed in the seventies of the last century by Moshinsky and Collins [8,9], whereas the special case with complex parameters was mentioned earlier by Bargmann [10]. Like the fractional Fourier transform (FrFT), LCT was used to solve differential equation and analyze optical system initially. With the development of FrFT in 1990s, LCT has been paid more and more attention in the field of signal processing [11,12]. The so-called Linear canonical Hilbert transform has been proposed to further generalize the HT [13,14]. The purpose of this paper is to study the Hilbert Transform in the LCT domain and its digital implementation.

II. HILBERT TRANSFORM IN THE LCT DOMAIN

A. Linear Canonical Transform

The LCT is defined as [1]

$$S_{(a,b,c,d)}(u) = S_{\mathbf{A}}(u) = \mathbf{L}^{\mathbf{A}}[s](u) = \mathbf{L}^{a,b,c,d}(s(t)) = \begin{cases} \sqrt{\dfrac{1}{j2\pi b}} \cdot \mathrm{e}^{\frac{jd}{2b}u^2} \displaystyle\int_{-\infty}^{+\infty} e^{-\frac{j}{b}ut} e^{\frac{ja}{2b}t^2} s(t)dt & b \neq 0 \\ \sqrt{d} \cdot \mathrm{e}^{\frac{jcd}{2}u^2} s(du) & b = 0 \end{cases}, ad - bc = 1, \quad (1)$$

where $\mathbf{A}$ denotes the parameters of LCT, $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

When $(a,b,c,d) = (\cos\alpha, \sin\alpha, -\sin\alpha, \cos\alpha)$, LCT can be simplified as

$$L^{(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha)}[s](u) = \sqrt{e^{-j\alpha}} S_\alpha(u), \quad (2)$$

where $S_\alpha(u)$ is the FrFT of $s(t)$ with the rotation angle α 。

When $(a,b,c,d) = (0,1,-1,0)$，LCT can be simplified as the Fourier transform (FT) with multiplier factor $\sqrt{-j}$ like

$$\begin{aligned} L^{(0,1,-1,0)}[s](u) &= \sqrt{-j} \cdot \mathbf{\Psi}[s](u) \\ &= \sqrt{-j} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} s(t) e^{-jut} dt = \sqrt{-j} \cdot S(u) \end{aligned}, \quad (3)$$

where $S(u)$ is the FT of $s(t)$, $\mathbf{\Psi}[\cdot]$ denotes the operator of FT.

B. Hilbert Transform

As early as 1946, Gabor applied Hilbert transform to analytic signal composition [1], which is as follows:

$$\tilde{s}(t) = s_{\mathrm{r}}(t) + js_{\mathrm{i}}(t), \quad (4)$$

where $s_{\mathrm{r}}(t)$ is a real signal, $\tilde{s}(t)$ denotes the analytical signal constructed on $s_{\mathrm{r}}(t)$, and $s_{\mathrm{i}}(t)$ is the Hilbert transform of $s_{\mathrm{r}}(t)$, i.e.

$$s_{\mathrm{i}}(t) = \mathbf{H}[s_{\mathrm{r}}](t), \quad (5)$$

where $\mathbf{H}[\cdot]$ denotes the Hilbert transformer. From the view of spectrum composition, the analytic signal is obtained by eliminating the negative spectrum part of the real signal. Since the FT is a kind of linear transformation, according to the construction equation of analytic signal as (4), the transfer function of Hilbert transform in the frequency domain can be obtained as follows:

$$H_{\mathrm{f}}(\omega) = -j\,\mathrm{sgn}(\omega) = \begin{cases} -j, & \omega \geq 0 \\ j, & \omega < 0 \end{cases}. \quad (6)$$

Therefore, following this mode, by eliminating the negative spectrum part of real signals in the LCT domain, the analytical signals in the LCT domain can be obtained as follows [4]:

$$\widetilde{s}_{\mathbf{A}}(t) = s(t) + j\mathbf{H}^{\mathbf{A}}[s](t), \quad b \neq 0, \tag{7}$$

where

$$\mathbf{H}^{\mathbf{A}}[s](t) = \frac{e^{-j\frac{d}{2b}t^2}}{-\pi}\int_{-\infty}^{+\infty}\frac{s(x)e^{j\frac{a}{2b}x^2}}{t-x}dx. \tag{8}$$

Equation (8) is the Hilbert transform in the LCT domain under the condition of analytic signal defined by (7). Further we can obtain

$$\mathbf{L}^{\mathbf{A}}\left[\mathbf{H}^{\mathbf{A}}[s]\right](u) = \begin{cases} -j\,\mathrm{sgn}(u)\mathbf{L}^{\mathbf{A}}[s](u), & \text{if } b<0 \\ j\,\mathrm{sgn}(u)\mathbf{L}^{\mathbf{A}}[s](u), & \text{if } b>0 \end{cases}. \tag{9}$$

Therefore, the transfer function of Hilbert transform in the LCT domain is as follows:

$$H_{\mathbf{A}}(u) = \begin{cases} -j\,\mathrm{sgn}(u), & \text{if } b<0 \\ j\,\mathrm{sgn}(u), & \text{if } b>0 \end{cases}. \tag{10}$$

Equation (9) can be arranged as

$$\mathbf{L}^{\mathbf{A}}\left[\mathbf{H}^{\mathbf{A}}[s]\right](u) = \mathbf{L}^{\mathbf{A}}[s](u)\cdot H_{\mathbf{A}}(u) = S_{\mathbf{A}}(u)\cdot H_{\mathbf{A}}(u). \tag{11}$$

III. DIGITAL IMPLEMENTATION

A. Implementation principle about Hilbert transform in the LCT domain

Based on the convolution theorem for the LCT [15], let $X_{\mathbf{A}}(u) = \mathbf{L}^{\mathbf{A}}[x](u)$, then we can obtain

$$\begin{aligned}\mathbf{L}^{\mathbf{A}}\left[x(t)*y(t)\right](u) &= \left|\frac{1}{a}\right| e^{j\frac{c}{2a}u^2}\left(\left(X_{\mathbf{A}}(u)e^{-j\frac{c}{2a}u^2}\right)*y\left(\frac{u}{a}\right)\right) \\ &= \left|\frac{1}{a}\right| e^{j\frac{c}{2a}u^2}\left(\breve{X}_{\mathbf{A}}(u)*y\left(\frac{u}{a}\right)\right)\end{aligned}, \tag{12}$$

where $a>0$. When $a<0$, the derivation has only one more coefficient, i.e. -1. Obviously

$$\mathbf{\Psi}\left[\breve{X}_{\mathbf{A}}(u)*y\left(\frac{u}{a}\right)\right](\omega) = \mathbf{\Psi}\left[\breve{X}_{\mathbf{A}}\right](\omega)\cdot aY(a\omega). \tag{13}$$

Let $\mathbf{\Psi}\left[\breve{X}_{\mathbf{A}}\right](u) = S_{\mathbf{A}}(u)$, $Y(au) = H_{\mathbf{A}}(u)$, $H_{\mathbf{A}}(u) = \mathbf{\Psi}[h_{\mathbf{A}}](u)$. we have

$$\mathbf{\Psi}\left[\breve{X}_{\mathbf{A}}\right](u)\cdot aY(au) = aS_{\mathbf{A}}(u)H_{\mathbf{A}}(u), \tag{14}$$

$$h_{\mathbf{A}}(t) = \mathbf{\Psi}^{-1}[H_{\mathbf{A}}](t) = \mathbf{\Psi}^{-1}[Y(au)](t) = \frac{1}{a}y\left(\frac{t}{a}\right), \tag{15}$$

where $\mathbf{\Psi}^{-1}[\cdot]$ denotes the inverse operator of the FT. Thus (13) can be manipulated as

$$\mathbf{\Psi}\left[\mathbf{\Psi}^{-1}[S_{\mathbf{A}}](t)*ah_{\mathbf{A}}(t)\right](\omega) = \mathbf{\Psi}\left[\breve{X}_{\mathbf{A}}\right](\omega)\cdot aY(a\omega). \tag{16}$$

Therefore

$$\begin{aligned}H_{\mathbf{A}}(u)\cdot S_{\mathbf{A}}(u) &= Y(au)\cdot\mathbf{\Psi}\left[\breve{X}_{\mathbf{A}}\right](u) \\ &= \mathbf{\Psi}\left[\mathbf{\Psi}^{-1}[S_{\mathbf{A}}](t)*h_{\mathbf{A}}(t)\right](u)\end{aligned}. \tag{17}$$

Let $\breve{X}_{\mathbf{A}}(u) = \mathbf{\Psi}^{-1}[S_{\mathbf{A}}](u)$, we replace t by u, we obtain

$$\begin{aligned}X_{\mathbf{A}}(u) &= \mathbf{\Psi}^{-1}[S_{\mathbf{A}}](u)\cdot e^{j\frac{c}{2a}u^2} \\ &= \frac{1}{\sqrt{j}}e^{j\frac{c}{2a}u^2}\cdot\mathbf{L}^{\mathbf{B}}\left[\mathbf{L}^{\mathbf{A}}[s]\right](u),\ \mathbf{B} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\end{aligned} \tag{18}$$

Thus

$$x(t) = \mathrm{real}\left[\mathbf{L}^{\tilde{\mathbf{A}}}[X_{\mathbf{A}}](t)\right], \tag{19}$$

where $\mathrm{real}[\cdot]$ denotes the operator for retaining the real part, $\tilde{\mathbf{A}} = \begin{pmatrix}d & -b\\ -c & a\end{pmatrix}$.

According to (12), then

$$e^{j\frac{c}{2a}u^2}\left(\mathbf{\Psi}^{-1}[S_{\mathbf{A}}](u)*h_{\mathbf{A}}(u)\right) = \mathbf{L}^{\mathbf{A}}\left[x(t)*h_{\mathbf{A}}(at)\right](u). \tag{20}$$

Let c=0, we obtain

$$\left(\mathbf{\Psi}^{-1}[S_{\mathbf{A}}](u)*h_{\mathbf{A}}(u)\right) = \mathbf{L}^{\mathbf{A}}\left[x(t)*h_{\mathbf{A}}(at)\right](u). \tag{21}$$

From (3), therefore

$$\begin{aligned}\mathbf{\Psi}\left[\mathbf{\Psi}^{-1}[S_{\mathbf{A}}](t)*h_{\mathbf{A}}(t)\right](u) &= \frac{\mathbf{L}^{\mathbf{B}}\left[\mathbf{L}^{\mathbf{A}}\left[x(t)*h_{\mathbf{A}}(at)\right]\right](u)}{\sqrt{-j}} \\ &= \mathbf{L}^{\mathbf{C}}\left[x(t)*\frac{1}{\sqrt{-j}}h_{\mathbf{A}}(at)\right](u)\end{aligned}, \tag{22}$$

where $\mathbf{B}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$,

$$\mathbf{C}=\mathbf{B}\cdot\mathbf{A}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\cdot\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}=\begin{pmatrix}0 & d\\ -a & -b\end{pmatrix}. \tag{23}$$

So we can find the implementation method of Hilbert transformer in the LCT domain with the parameters as **C**. This method is described as follows:

Step 1: based on $s(t)$ and **C**, **A**, $x(t)$ and $h_{\mathbf{A}}(t)$ are obtained from (23), (19), (10), and (15).

Step 2: through the convolution of $x(t)$ and $\frac{1}{\sqrt{-j}}h_{\mathbf{A}}(at)$.

we can obtain the Hilbert transform of $s(t)$ in the LCT domain with the parameters, **C**.

B. Digital implementation associated with FIR filter

The above method implements Hilbert transform in LCT domain. However, engineering applications are often based on digital signal processing. Therefore, it is necessary to further obtain the digital implementation of Hilbert transform in the LCT domain. In other words, the above method is needed to be converted to discrete Hilbert transform in the LCT domain. So the direct way is to use the transfer function of Hilbert transform in the LCT domain shown in (10). First, $S_{\mathrm{A}}(m)$ is obtained through the discrete LCT of $s(n)$. Then, the discrete LCT of $s(n)$ is obtained via the multiplication of $S_{\mathrm{A}}(m)$ and the discrete version of the transfer function $H_{\mathbf{A}}(u)$, as shown as (10). This route is plausible, but this is not , in fact, feasible. Because the periodic continuity will affect the HT in the LCT domain, when processing in the analog mode is switched to processing in the digital mode. That is to say, the transfer function $H_{\mathbf{A}}(u)$, as shown as (10), will be transformed into the periodic version $H_{\mathrm{d}\mathbf{A}}(u)$, as shown as Fig. 1. In Fig. 1, $f_{\mathrm{s}}=1/t_{\mathrm{s}}$ denotes the sampling frequency, $b\in\mathbf{A}$ and $b<0$.

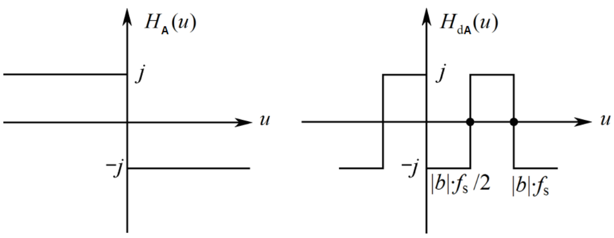

Fig. 1 Transfer function of Hilbert transform in the LCT domain (when $b<0$).

From Fig. 1, it can be found that the transfer function $H_{\mathrm{d}\mathbf{A}}(u)$ is an odd function, whose period is $|b|\cdot f_{\mathrm{s}}$ [15]. Thus, it can be expanded by the Fourier series as

$$H_{\mathrm{d}\mathbf{A}}(u)=\sum_{n=1}^{+\infty}a_n\sin\left(2\pi nut_{\mathrm{s}}/|b|\right), \tag{24}$$

$$\begin{aligned}a_n&=\frac{2}{|b|\cdot f_{\mathrm{s}}}\int_{-|b|\cdot f_{\mathrm{s}}/2}^{|b|\cdot f_{\mathrm{s}}/2}H_{\mathrm{d}\mathbf{A}}(u)\cdot\sin\left(2\pi nut_{\mathrm{s}}/|b|\right)du\\&=\begin{cases}0 & ,\ n \text{ is even}\\ -\frac{4j}{n\pi} & ,\ n \text{ is ood}\end{cases}\end{aligned}. \tag{25}$$

The response in the time domain , $h_{\mathrm{d}\mathbf{A}}(t)$, can be discreted as $h_{\mathrm{d}\mathbf{A}}(nt_{\mathrm{s}})$. So the transform equation of the discrete HT in the LCT domain is as follows:

$$s_{\mathrm{di}}(nt_{\mathrm{s}})=\mathbf{H}^{\mathrm{dA}}\left[s_{\mathrm{dr}}\right](nt_{\mathrm{s}})=s_{\mathrm{dr}}(nt_{\mathrm{s}})*h_{\mathrm{d}\mathbf{A}}(nt_{\mathrm{s}}), \tag{26}$$

where $s_{\mathrm{dr}}(nt_{\mathrm{s}})$ and $s_{\mathrm{di}}(nt_{\mathrm{s}})$ are these discrete samples of $s_{\mathrm{r}}(t)$ and $s_{\mathrm{i}}(t)$ respectively. Convolution can be understood as the superposition of input signal $s_{\mathrm{dr}}(nt_{\mathrm{s}})$ weighted by the response in the time domain $h_{\mathrm{d}\mathbf{A}}(nt_{\mathrm{s}})$, so it can be realized through the transverse delay filter. But first you need to determine the response in the time domain, $h_{\mathrm{d}\mathbf{A}}(nt_{\mathrm{s}})$.

Z-transformation of $h_{\mathrm{d}\mathbf{A}}(nt_{\mathrm{s}})$ is as follows:

$$\begin{aligned}H_{\mathrm{d}\mathbf{A}}(Z)&=\sum_{n=-\infty}^{+\infty}h_{\mathrm{d}\mathbf{A}}(nt_{\mathrm{s}})Z^{-n}\\&=h_{\mathrm{d}\mathbf{A}}(0)+\sum_{n=1}^{+\infty}\left[h_{\mathrm{d}\mathbf{A}}(-nt_{\mathrm{s}})Z^{n}+h_{\mathrm{d}\mathbf{A}}(nt_{\mathrm{s}})Z^{-n}\right]\end{aligned}. \tag{27}$$

Using the variable substitution $Z=e^{j2\pi ft_{\mathrm{s}}}$, and launch from Euler's formula, we have

$$\begin{aligned}&H_{\mathrm{d}\mathbf{A}}(f)=h_{\mathrm{d}\mathbf{A}}(0)+\\&\sum_{n=1}^{+\infty}\left[h_{\mathrm{d}\mathbf{A}}(-nt_{\mathrm{s}})\cos(2\pi nft_{\mathrm{s}})+jh_{\mathrm{d}\mathbf{A}}(-nt_{\mathrm{s}})\sin(2\pi nft_{\mathrm{s}})\right.\\&\left.+h_{\mathrm{d}\mathbf{A}}(nt_{\mathrm{s}})\cos(2\pi nft_{\mathrm{s}})-jh_{\mathrm{d}\mathbf{A}}(nt_{\mathrm{s}})\sin(2\pi nft_{\mathrm{s}})\right]\end{aligned}. \tag{28}$$

In the form of real part and imaginary part, (28) is written as

$$H_{\mathrm{d}\mathbf{A}}(f)=H_{\mathrm{rd}\mathbf{A}}(f)+\mathrm{j}H_{\mathrm{id}\mathbf{A}}(f), \tag{29}$$

where

$$H_{\mathrm{rd}\mathbf{A}}(f) = h_{\mathrm{d}\mathbf{A}}(0) + \sum_{n=1}^{+\infty}\left[h_{\mathrm{d}\mathbf{A}}(-nt_{\mathrm{s}}) + h_{\mathrm{d}\mathbf{A}}(nt_{\mathrm{s}})\right]\cos(2\pi nft_{\mathrm{s}})$$
$$H_{\mathrm{id}\mathbf{A}}(f) = \sum_{n=1}^{+\infty}\left[h_{\mathrm{d}\mathbf{A}}(-nt_{\mathrm{s}}) - h_{\mathrm{d}\mathbf{A}}(nt_{\mathrm{s}})\right]\sin(2\pi nft_{\mathrm{s}}) \quad (30)$$

Since $H_{\mathrm{d}\mathbf{A}}(e^{j2\pi ft_{\mathrm{s}}})$ has only imaginary parts but no real parts, we have $h_{\mathrm{d}\mathbf{A}}(0)=0$, and $h_{\mathrm{d}\mathbf{A}}(-nt_{\mathrm{s}})=-h(nt_{\mathrm{s}})$. Thus,

$$H_{\mathrm{d}\mathbf{A}}(f) = -2j\sum_{n=1}^{+\infty}\left[\left[h_{\mathrm{d}\mathbf{A}}(nt_{\mathrm{s}})\sin(2\pi nft_{\mathrm{s}})\right]\right] \quad (31)$$

Since $u=b\cdot f$, then (24) can be manipulated as

$$H_{\mathrm{d}\mathbf{A}}(f) = \sum_{n=1}^{+\infty} -a_n \sin(2\pi nft_{\mathrm{s}}), \quad (32)$$

According to (25), we have

$$h_{\mathrm{d}\mathbf{A}}(nt_{\mathrm{s}}) = \begin{cases} 0 & , n \text{ is even} \\ -\dfrac{2}{n\pi}, & n \text{ is ood} \end{cases}. \quad (33)$$

From (33), then

$$\frac{1}{\sqrt{-j}} h_{\mathrm{d}\mathbf{A}}(nat_{\mathrm{s}}) = \begin{cases} 0 & , n \text{ is even} \\ -\dfrac{2\sqrt{ja}}{n\pi}, & n \text{ is ood} \end{cases}. \quad (34)$$

Thus, we can propose the two steps for implementing the discrete HT of $s(t)$ in the LCT domain with the parameters C, which is as follows:

Step 1: based on $s(nt_{\mathrm{s}})$ and **C**, we can obtain **A**, $x(nt_{\mathrm{s}})$ from (23), (19), (10), and (15).

Step 2: from (26) and (34), the discrete convolution of $x(nt_{\mathrm{s}})$ and $\frac{1}{\sqrt{-j}} h_{\mathrm{d}\mathbf{A}}(nat_{\mathrm{s}})$ is realized through FIR filter.

IV. Conclusions

Since the LCT is the generalization of FT, we follow the route that the analytic signal is obtained by the classic Hilbert transform, and constitute the analytic signal for the LCT to eliminate the negative parts of the LCT spectrum. Further we studied the HT in the LCT domain, and propose the digital implementation method.

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References

[1] D.Gabor, “Theory of Communication,” *J.Inst.EE 93(III)*, pp. 429-457, 1946.

[2] R. E. Ziemer and W. H. Traner, *Principles of Communications-Systems, Modulation, and Noise*. Boston, MA:Houghton Mifflin, 1990.

[3] S. Hahn, *Hilbert Transforms in Signal Processing*. Boston, MA:Artech House, 1996.

[4] A. Venjutatanab, C.S. Seelamantula, “Fractional Hilbert transform extensions and associated analytic signal construction,” *Signal Process.*, vol. 94, pp. 359-372, 2014.

[5] Chien-Cheng Tseng. Su-Ling Lee, “Wideband digital fractional Hilbert transformer designs based on interlaced and derivative sampling methods,” *Signal Process.*, vol. 123, pp. 84-99, 2016.

[6] Ran Tao, Xuemei Li, and Yue Wang, “Generalization of the fractinal Hilbert transform,” *IEEE Signal Process. Lett.*, vol. 15, pp. 365-368, 2008.

[7] A. Zayed, “Hilbert transform associated with the fractional Fourier transform,” *IEEE Signal Process. Lett.*, vol. 5, pp. 206-208, 1998.

[8] M. Moshinsky and C. Quesne, “Linear canonical transformations and their unitary representations,” *J.Math.Phys.*, vol. 12, no.8, pp. 1772-1780, 1971.

[9] S. A. Collins, “Lens-system diffraction integral written in terms of matrix optics,” *J.Opt.Soc.Am.*, vol. 60, pp. 1168-1177, 1970.

[10] V. Bargmann, “On a Hilbert space of analytic functions and an associated integral transform, Part I,” *Comm. Pure. Appl. Math.*, vol. 14, pp. 187-214, 1961.

[11] R. Tao, B. Deng, Y. Wang, F*ractional Fourier transform and its applications*. Beijing: Tsinghua University Press, 2009.

[12] Ding Jian-Jiun. *Research of Fractional Fourier Transform and Linear Canonical Transform*. Taipei: Department of Electrical Engineering, National Taiwan University, 2000.

[13] S Xu, C Yi, Y Hu, et al, “Uncertainty inequalities for the linear canonical Hilbert transform,” *Circuits System & Signal Processing*, vol. 3, pp. 1-15, 2018.

[14] SC Pei, SG Huang, “Reversible joint Hilbert and linear canonical transform without distortion,” *IEEE Transactions on Sginal Processing*, vol. 61, no. 19, pp. 4768-4781, 2013.

[15] Bing Deng, Ran Tao, Yue Wang, “Convolution theorems for the linear canonical transform and their applications,” *Science in China Series F: Information Sciences*, vol. 49, pp. 592-603, 2006.